

Completion of the mixed unit interval graphs hierarchy

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Abstract

We describe the missing class of the hierarchy of mixed unit interval graphs, generated by the intersection graphs of closed, open and one type of half-open intervals of the real line. This class lies strictly between unit interval graphs and mixed unit interval graphs. We give a complete characterization of this new class, as well as quadratic-time algorithms that recognize graphs from this class and produce a corresponding interval representation if one exists. We also mention that [8] directly extends to provide a quadratic-time algorithm to recognize the class of mixed unit interval graphs.

Keywords: unit interval graph; mixed unit interval graph; proper interval graph; intersection graph

1 Introduction

A graph is an interval graph if one can associate with each of its vertices an interval of the real line such that two vertices are adjacent if and only if the corresponding intervals intersect. A well-studied subclass of the class of interval graphs is the one of proper interval graphs, that is when we require that no interval properly contains another one. This class coincides with the class of unit interval graphs (when all intervals have length one [7]).

However, in the previous descriptions no particular attention is paid to the types of intervals that are used: are they open, closed, or semi-closed? Dourado, Le, Protti, Rautenbach and Szwarcfiter proved in [1] that this is of no importance as far as interval graphs are concerned. However, this is not true for unit interval graphs: deciding which types of intervals are allowed to represent the vertices of a graph is crucial. This fact was notably studied in [7], [2], [6], [1], [3] and [8]. In these papers one can find results about the classes of graphs we can get depending on the types of unit intervals we allow for their representations. In particular, it is shown that if we require all the unit intervals used for representing a graph to be of the same type (either all closed, all open, all left-closed-right-open, or all left-open-right-closed), one gets the same class of *unit interval graphs*. This class is a proper subclass of *mixed unit interval graphs*, *i.e.* graphs obtained when the intervals are only required to be of unit length. Recently, Joos [3] gave a characterization of mixed unit interval graphs by an infinite class of forbidden induced subgraphs, and Shuchat, Shull, Trenk and West [8] complemented it by a quadratic-time algorithm

which, given any graph in this class, outputs an corresponding mixed-unit interval representation. In [5], Le and Rautenbach took a different approach and studied the graphs which are representable by intervals beginning at integer positions.

The aim of this paper is to complete this hierarchy of classes. We consider all subsets of the four types of unit intervals, show that several of them lead to the classic unit interval graphs (when all intervals are closed and of length one), recall the previously studied and characterized class determined by open and closed unit intervals, and then show that – with respect to this parametrization – there exists exactly one more proper subclass of the class of mixed unit interval graphs. We characterize this class by an infinite list of forbidden induced subgraphs and give quadratic-time algorithms that check whether a graph belongs to this class, and in case it does, produce a corresponding appropriate interval representation.

2 Preliminaries

2.1 Basic definitions and notations

All the graphs we consider here are finite, undirected, and simple. Let G be a graph. We denote the vertex and edge set of G by $V(G)$ and $E(G)$, respectively, or V and E if there are no ambiguities. We say that two vertices u and v are neighbors, adjacent, or connected if $\{u, v\} \in E(G)$.

For a vertex $v \in V(G)$, let the *neighborhood* $N_G(v)$ of v be the set of all vertices which are adjacent to v and let the *closed neighborhood* $N_G[v]$ be defined as $N_G(v) \cup \{v\}$. Two distinct vertices u and v are *twins* (in G) if $N_G[u] = N_G[v]$. If G contains no twins, then G is *twin-free*.

If C is a set of vertices, then we denote by $G[C]$ the subgraph of G induced by C .

Let \mathcal{M} be a set of graphs. We say that G is \mathcal{M} -free if for every $H \in \mathcal{M}$, the graph H is not an induced subgraph of G .

Let \mathcal{N} be a family of intervals. We say that a graph G has an \mathcal{N} -representation if there is a function $I : V(G) \rightarrow \mathcal{N}$ such that any two distinct vertices u and v are adjacent if and only if $I(u) \cap I(v) \neq \emptyset$. We say that G is an \mathcal{N} -graph if there is an \mathcal{N} -representation of G .

Let $x, y \in \mathbb{R}$. We define the *closed interval* $[x, y] = \{z \in \mathbb{R} : x \leq z \leq y\}$, the *open interval* $(x, y) = \{z \in \mathbb{R} : x < z < y\}$, the *closed-open interval* $[x, y) = \{z \in \mathbb{R} : x \leq z < y\}$ and the *open-closed interval* $(x, y] = \{z \in \mathbb{R} : x < z \leq y\}$. All along this paper we draw the different types of intervals as follows:



Figure 1: The closed, open, closed-open, and open-closed intervals.

For an interval A , let $\ell(A) = \inf(\{x \in \mathbb{R} : x \in A\})$ and $r(A) = \sup(\{x \in \mathbb{R} : x \in A\})$. If I is an interval representation of G and $v \in V(G)$, then we write $\ell(v)$ and $r(v)$ instead of $\ell(I(v))$ and $r(I(v))$, if there are no ambiguities.

Let \mathcal{U}^{++} be the set of all closed unit intervals of the real line, \mathcal{U}^{--} be the set of all open unit intervals, \mathcal{U}^{-+} be the set of all open-closed unit intervals, \mathcal{U}^{+-} be the set of all closed-open unit intervals, and \mathcal{U} be the set of all unit intervals. We also define $\mathcal{U}^{\pm} = \mathcal{U}^{++} \cup \mathcal{U}^{--}$ and $\mathcal{U}^X = \bigcup_{x \in \{X\}} \mathcal{U}^x$ when $\{X\} \subseteq \mathcal{P}(\{++, --, -+, +-, \pm\})$. For instance, $\mathcal{U} = \mathcal{U}^{\pm, +-, -+}$. In this terminology, \mathcal{U} -graphs are *mixed unit interval graphs*. Let us also call a $\mathcal{U}^{\pm, +-, -+}$ -graph an *almost-mixed unit interval graph*.

2.2 Previous results

First we can see that if a graph contains twins, then they can be assigned the same intervals, so in what follows we will mostly consider twin-free graphs. We will denote by \mathcal{G}^X the set of all twin-free \mathcal{U}^X -graphs.

We begin by recalling the known results on classifying and characterizing the unit interval graph classes. The following two theorems characterize completely the most simple one:

Theorem 1 (Roberts [7]). *A graph G is a \mathcal{U}^{++} -graph if and only if it is a $K_{1,3}$ -free interval graph.*

Theorem 2 (Dourado et al. [1], Frankl and Maehara [2]). *The classes of \mathcal{U}^{++} -graphs, \mathcal{U}^{--} -graphs, \mathcal{U}^{+-} -graphs, \mathcal{U}^{-+} -graphs, and $\mathcal{U}^{+-,-+}$ -graphs are the same.*

The next theorem characterizes the set of twin-free graphs of the class of \mathcal{U}^\pm -graphs, that is when we allow both closed and open intervals but no others. This class lies just above the class of \mathcal{U}^{++} -graphs.

Theorem 3 (Rautenbach and Szwarcfiter [6]). *A graph G is in \mathcal{G}^\pm if and only if G is a $\{K_{1,4}, K_{1,4}^*, K_{2,3}^*, K_{2,4}^*\}$ -free interval graph.*

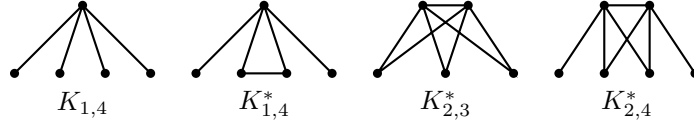


Figure 2: Forbidden induced subgraphs for twin-free \mathcal{U}^\pm -graphs

It is easy to see that these three previous classes of interval graphs are not the same. Indeed, $K_{1,3}$ is a \mathcal{U}^\pm -graph but not a \mathcal{U}^{++} -graph. Also, the graph of Figure 3 is a \mathcal{U} -graph but not a \mathcal{U}^\pm -graph. A characterization of twin-free \mathcal{U} -graphs was recently given by Joos (the classes \mathcal{R} , \mathcal{S} , \mathcal{S}' , and \mathcal{T} of forbidden induced subgraphs are depicted in Figures 4–7):



Figure 3: A graph, which is a \mathcal{U} -graph, but not a \mathcal{U}^\pm -graph

Theorem 4 (Joos [3]). *A graph G is in \mathcal{G} if and only if G is a $\{K_{2,3}^*\} \cup \mathcal{R} \cup \mathcal{S} \cup \mathcal{S}' \cup \mathcal{T}$ -free interval graph.*



Figure 4: The class \mathcal{R}

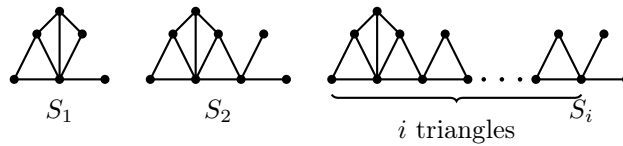


Figure 5: The class \mathcal{S}

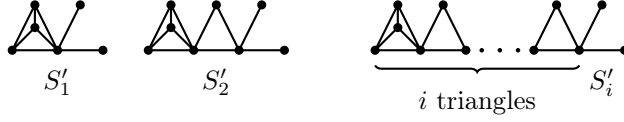


Figure 6: The class \mathcal{S}'

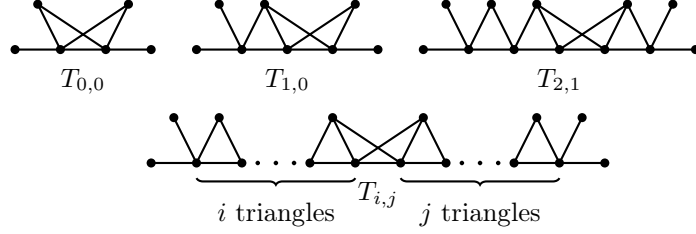


Figure 7: The class \mathcal{T}

To summarize, so far we have the following inclusions, all being proper:

$\{(\emptyset, \emptyset)\} \subsetneq \{\mathcal{U}^{++}, \mathcal{U}^{--}, \mathcal{U}^{+-}, \mathcal{U}^{-+}, \text{ or } \mathcal{U}^{+-, -+}\}\text{-graphs} \subsetneq \mathcal{U}^{\pm}\text{-graphs} \subsetneq \mathcal{U}\text{-graphs}.$

However so far we have seen only 9 different sets of unit interval types, out of the 16 which exist. In the next section we will complete the picture.

3 Our results

In this part we take care of each of the 7 missing subsets for the unit interval representations of graphs. We first consider the subsets which lead to the class of \mathcal{U}^{++} -graph, and then introduce the new class of almost-mixed unit interval graphs.

3.1 Completion of the unit interval graphs hierarchy

Theorem 5. *The classes of \mathcal{U}^{++} -graphs, $\mathcal{U}^{++,-}$ -graphs, $\mathcal{U}^{++,-+}$ -graphs, $\mathcal{U}^{--,-}$ -graphs, $\mathcal{U}^{--,-+}$ -graphs, $\mathcal{U}^{+-,-}$ -graphs and $\mathcal{U}^{+-,-+}$ -graphs are the same.*

Proof. The proof is simple. Firstly each of these classes contains the class of \mathcal{U}^{++} -graphs.

Secondly, $K_{1,3}$, which is the only minimal forbidden induced subgraph for \mathcal{U}^{++} -graphs, is in none of these classes. Indeed, let us draw a unit interval representation of $K_{1,3}$ and show that we then need both closed and open intervals to do so. We label the vertices as in Figure 8. We may assume, without loss of generality, that $\ell(c) = 0$ and that $\ell(a) \leq \ell(b) \leq \ell(d)$. All intervals having length one, their intersections enforce the following inequality: $1 = \ell(c) + 1 \geq \ell(d) \geq \ell(b) + 1 \geq \ell(a) + 2 \geq \ell(c) + 1 = 1$. This forces $\ell(a) = -1$, $\ell(b) = 0$ and $\ell(d) = 1$. It follows that $I(c)$ must be a closed interval, the right end of $I(a)$ must be closed and the left end of $I(d)$ must be closed too. To meet the required intersections, $I(b)$ must have open ends, which concludes the proof. ■



Figure 8: The “claw” $K_{1,3}$ and its unique \mathcal{U} -representations

We now deal with the remaining two subsets of intervals $\mathcal{U}^{\pm,+}$ and $\mathcal{U}^{\pm,-}$ which lead, by symmetry, to the same class of graphs. We first show that this is a proper new class. In order to do so, we introduce a lemma about the essence of the $\mathcal{U}^{\pm,+}$ class: the existence of an induced $K_{1,4}^*$.

We call a representation *injective* if no two vertices are represented by the same interval. Note that every representation of a twin-free graph is injective.

Lemma 1. *Up to symmetry, there are only two injective \mathcal{U} -representations of $K_{1,4}^*$, shown in Figure 9 (the leftmost interval is either open-closed or closed).*

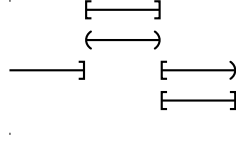


Figure 9: The unique injective representations of $K_{1,4}^*$

Proof. Let us consider I an injective \mathcal{U} -representation of $K_{1,4}^*$. From the proof of Theorem 5, we can see that every $K_{1,3}$ must be represented this way:

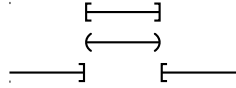


Figure 10: The unique injective \mathcal{U} -representations of $K_{1,3}$

Let us denote the two vertices of degree one of $K_{1,4}^*$ by a and b , the vertex of maximum degree by c , and the other two nodes by d and e . We have the following claws: $cabd$ and $cabe$. Since c is connected to all the other vertices, $I(c)$ must be the middle closed interval in Figure 10. Then $I(a)$ and $I(b)$ cannot be the extremal intervals, else there would be no intervals for both d and e : I being injective, they must be assigned distinct intervals. Therefore for instance $I(a)$ is the leftmost interval and $I(b)$ is the middle open one. Now d and e must both be at the position of the rightmost interval. Since I is injective, one of them must be closed-open and the other one closed. Note that the left end of the leftmost interval is free. ■

Theorem 6. *The following strict inclusions hold: \mathcal{U}^{\pm} -graphs $\subsetneq \mathcal{U}^{\pm,+}$ -graphs $\subsetneq \mathcal{U}$ -graphs.*

Proof. The inclusions are immediate, we only need to show that they are strict.

First we give a $\mathcal{U}^{\pm,+}$ -representation of the graph in Figure 3, which is not a \mathcal{U}^{\pm} -graph.

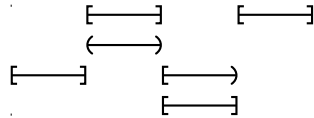


Figure 11: A $\mathcal{U}^{\pm,+}$ -representation of the graph of Figure 3

Now we show in Figure 12 a graph which is a \mathcal{U} -graph, but not a $\mathcal{U}^{\pm,+}$ one.

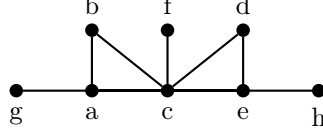


Figure 12: A graph separating $\mathcal{U}^{\pm,+-}$ -graphs and \mathcal{U} -graphs

Let us draw an injective \mathcal{U} -representation of this graph, and show that it is unique up to a few changes. We will see that this representation needs all four types of intervals, hence our result. First we can see that it contains two induced $K_{1,4}^*$: $cfeab$ and $cf dab$. By Lemma 1 and the fact that f is only adjacent to c , $I(c)$ and $I(f)$ are completely determined as in Figure 13. Now given the neighborhood of a, b, d and e , and the fact that both a and e have one neighbor which is not adjacent to any other vertex, the intervals of a, b, d and e are again completely determined, up to symmetry, as in Figure 13. This shows that we need all four types of intervals to draw this graph. ■

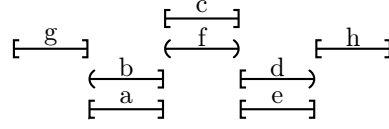


Figure 13: A \mathcal{U} -representation of the graph in Figure 12

To conclude this part, we now have a complete picture of the different subclasses of the mixed unit interval class. In the schematic Figure 14, $\mathcal{U}^X \subsetneq \mathcal{U}^Y$ is a shorthand notation for \mathcal{U}^X -graphs $\subsetneq \mathcal{U}^Y$ -graphs. Sets separated by commas define the same classes of graphs.

$$\begin{aligned}
& \emptyset \\
& \mathcal{U}^{\pm} \\
& \mathcal{U}^{++}, \mathcal{U}^{--}, \mathcal{U}^{+-}, \mathcal{U}^{-+}, \mathcal{U}^{+-,-+}, \mathcal{U}^{++,-+}, \mathcal{U}^{+-,+}, \\
& \mathcal{U}^{--,-+}, \mathcal{U}^{+-,-+}, \mathcal{U}^{++,-+,-+}, \mathcal{U}^{--,-+,-+} \\
& \mathcal{U}^{\pm} \\
& \mathcal{U}^{\pm} \\
& \mathcal{U}^{\pm} \\
& \mathcal{U}^{\pm,+-}, \mathcal{U}^{\pm,-+} \\
& \mathcal{U}^{\pm} \\
& \mathcal{U}
\end{aligned}$$

Figure 14: Classification of the subclasses of the mixed unit interval graphs

3.2 Characterization of the new class

In this part, we characterize the new class of twin-free almost-mixed unit interval graphs, $\mathcal{G}^{\pm,+-}$, by a list of minimal forbidden induced subgraphs. We begin by finding this list through a reasoning by inference, and afterwards check that all these graphs are indeed forbidden, and minimal.

We recall that since the graphs in \mathcal{G} are twin-free, any representation of such a graph is injective.

We first present a lemma which will prove to be very important in what follows. It guarantees that any graph belonging to \mathcal{G} has a “good” interval representation in which each half-open interval is eventually surrounded by a certain neighborhood of intervals.

Definition 1. Let $G \in \mathcal{G}$ and I be a mixed-unit interval representation of G . Let $\alpha(I)$ (resp. $\beta(I)$) be the number of open-closed (resp. closed-open) intervals in I . We say that I is *minimal* if the couple $(\alpha(I), \beta(I))$ is lexicographically minimal among all other representations of G , that is if I' represents G then either $\alpha(I') > \alpha(I)$ or $\alpha(I') = \alpha(I)$ and $\beta(I') \geq \beta(I)$.

Lemma 2. Let $G \in \mathcal{G}$ and I be a minimal \mathcal{U} -representation of it. Then, for every vertex \hat{u} (resp. \hat{d}) such that $I(\hat{u})$ (resp. $I(\hat{d})$) is an open-closed (resp. closed-open) interval, there exist vertices u, v, w, x, y (resp. a, b, c, d, e) in the same connected component as \hat{u} (resp. \hat{d}) such that their intervals are the following:



Proof. We prove the lemma only for the case of \hat{u} , the other one being completely symmetrical. Also, up to translation, we will assume that $\ell(\hat{u}) = 0$. We will proceed by contradiction, assuming that one interval is missing, which will enable us to contradict the minimality of I .

The overall idea of the proof is that, if one of the mentioned intervals is missing, then we can shift some intervals and close the left end of $I(u)$ so as to get a representation I' , equivalent to I , with the same number of closed-open intervals but with one fewer open-closed intervals, hence a contradiction. To do so, we first define

$$\varepsilon = \min(\{1\} \cup \{|x - y| : x, y \in \bigcup_{t \in V(G)} \{l(t), r(t)\} \wedge x \neq y\}).$$

This quantity equals the smallest *non-zero* distance between any endpoints of any two intervals, or 1 if the graph is empty. We will use it as a security distance: it guarantees that, given an extremity of any interval, other intervals can end at the same point or at least ε away from this point.

We begin by two useful remarks:

Remark 1. Let $0 < \varepsilon' < \varepsilon$. If a vertex x is such that $I(x)$ has an open left (resp. right) end, we can either shift it by ε' (resp. $-\varepsilon'$) or shift any other set of intervals by $-\varepsilon'$ (resp. ε') without losing any intersection involving $I(x)$ (but we can gain intersections).

This comes from the definition of ε : since the left end of $I(x)$ is open, any interval intersecting it at its left must do it on more than a single point, hence the intersection is of length at least $\varepsilon > \varepsilon'$.

Definition 2. We say that the interval of a vertex x is *left-free* (resp. *right-free*) if there is no other vertex t such that $r(t) = \ell(x)$ (resp. $\ell(t) = r(x)$).

Remark 2. Let $I(x)$ be a left-free (resp. right-free) interval. Closing its left (resp. right) end does not create any intersection.

Definition 3. We say that a vertex x has an *integer interval* if $\ell(x) \in \mathbb{Z}$.

Claim 1. If $I(\hat{u})$ is open-closed, then there exists some closed $I(\hat{v})$ at the same position.

Proof of Claim 1. We suppose to the contrary that there is no such $I(\hat{v})$. We would like to close the left end of $I(\hat{u})$. To do so, let us define I' in the following way:

- $I'(t) = I(t) - \varepsilon/2$ if $t \neq \hat{u}$, $\ell(I(t)) \in \mathbb{Z}$ and $\ell(I(t)) \leq 0$;
- $I'(\hat{u}) = [0, 1]$ (now it is closed);
- $I'(t) = I(t)$, otherwise.

We now show that I and I' are equivalent. By the definition of ε , we modify no intersection involving any non-integer interval. Since we do not shift the intervals beginning from 1 on, and we shift all integer intervals $J \in I(G)$ such that $\ell(J) \leq 0$ by the same quantity, the only intersections we can change involve $I(\hat{u})$ or an interval at the same position as $I(\hat{u})$. Since I is injective and there is no $[0, 1]$ interval, any interval sharing the position of $I(\hat{u})$ must have an open right end. Therefore, it had no intersection at 1, and shifting it does not remove any intersection. The same applies for $I(\hat{u})$: since its left end is open, it does not lose any intersection. Moreover, since we shifted all other integer intervals, we can close it without creating any new intersection.

This shows the equivalence between I and I' , which contradicts the minimality of I . □

Claim 2. *If $I(\hat{u})$ is open-closed, then there exists some closed $I(\hat{w})$ as in the statement of the lemma.*

Proof of Claim 2. We again proceed by contradiction, and suppose that no such interval exists. We define I' as follows:

- $I'(t) = I(t) - \varepsilon/2$ if $t \neq \hat{u}$, $\ell(I(t)) \in \mathbb{Z}$ and $\ell(I(t)) \leq 0$;
- $I'(\hat{u}) = [-\varepsilon/4, 1 - \varepsilon/4]$;
- $I'(t) = I(t)$, otherwise.

Using the same arguments as in Claim 1, we conclude that the first line of the definition of I' preserves all the intersections and creates none, except possibly the ones with $[1, 2]$ or $[1, 2)$. However, by assumption there is no $[1, 2]$ interval and then by the contrapositive of the “*abcde*” version of Claim 1, there is no $[1, 2)$ interval, so in I' \hat{u} also keeps exactly the intersections it has in I . For the same reason, shifting $I(\hat{u})$ by $-\varepsilon/4$ removes no intersections at its right. Since we shift it by less than the other intervals, it is now left-free, and so Remark 2 guarantees that by closing its left end we create no intersection.

Therefore, I and I' are equivalent, which is a contradiction to the minimality of I . □

Claim 3. *If $I(\hat{u})$ is open-closed, then there exist, in the same connected component as \hat{u} , some vertices u, v, w and y with intervals as in the lemma and such that there is no open-closed interval at the same position as $I(\hat{y})$.*

Proof of Claim 3. We assume that $I(\hat{u}) = [0, 1]$. From the previous two claims, we may assume that we have some vertices \hat{v}, \hat{w} such that $I(\hat{v}) = [0, 1]$ and $I(\hat{w}) = [1, 2]$.

We first suppose that there exists neither such a $(1, 2)$ interval, nor a $(1, 2]$ interval. We then define I' as follows:

- $I'(\hat{u}) = [\varepsilon/2, 1 + \varepsilon/2]$;
- $I'(t) = I(t)$, otherwise.

The interval representations I and I' are equivalent: since there is no interval with an open left end at 1, shifting $I(\hat{u})$ does not make it gain any intersection. By Remark 1, it loses none at its left. Furthermore, by definition of ε , $I(\hat{u}) + \varepsilon/2$ is left-free, so by Remark 2 we can close it without adding any intersection. This contradicts the minimality of I . Therefore, we may assume that there exists at least one of these two intervals: $(1, 2)$ or $(1, 2]$. But if it is the latter, then we can choose instead $\hat{u} = (1, 2]$ and restart the proof with the new \hat{u} . The number of intervals being finite, at the end we get an open interval $I(\hat{y})$ as we want. We choose the last set of such vertices for u, v, w and y .

We notice that by our choice of $I(y)$ there is no open-closed interval at the same position. In addition, because of the connection of our intervals, y is in the same connected component as \hat{u} . \square

We now show the existence of $I(z)$. Because of the previous claims, we know that there exist vertices u, v, w and y with the intervals we want. We now assume that $\ell(u) = 0$.

We proceed again by contradiction: if there were no such $[2, 3]$ interval, then we can define I' as follows:

- $I'(t) = I(t) + \varepsilon/2$ if $\ell(I(t)) \in \mathbb{Z}$, $\ell(I(t)) \geq 2$;
- $I'(y) = (1 + \varepsilon/2, 2 + \varepsilon/2)$;
- $I'(u) = [\varepsilon/2, 1 + \varepsilon/2]$;
- $I'(t) = I(t)$, otherwise.

We show that I and I' are equivalent. Since there is no $[2, 3]$ interval, by the contrapositive of Claim 1 there is no $[2, 3)$ interval, hence by the same arguments as in the proof of Claim 1, we lose no intersection by the first line of the definition of I' . Owing to the first shift and the definition of ε , shifting $I(y)$ does not create any intersection at its right. Since its left end is open, Remark 1 guarantees that we lose no intersection at its left. Since $I(u)$ has an open left end, shifting it modifies no intersection at its left. Since we have shifted $I(y)$ and there is no $(1, 2]$ interval, we create no intersection at its right. Besides, $I'(u)$ is now left-free, hence we can close it. This contradicts again the minimality of I . \blacksquare

Now we look for all possible forbidden induced minimal subgraphs of any $G \in \mathcal{G} \setminus \mathcal{G}^{\pm,+-}$. Let us take such a graph G and consider I a *minimal* \mathcal{U} -representation of G , that is one *with minimum number of open-closed intervals, and subject to this condition, minimum number of closed-open intervals*.

First, since $G \notin \mathcal{G}^{\pm,+-}$, there exist one open-closed interval $I(u)$ and one closed-open interval $I(d)$ which, by Lemma 2, come with some neighbors a, b, c, e, v, w, y, z represented by intervals exactly as in the lemma.

Remark 3. We may assume that $I(u)$ and $I(d)$ are connected through a succession of intervals.

Proof. We proceed by contradiction. If every such pair (u, d) was composed of vertices in different connected components, then by symmetrizing the interval representation of all components containing (only) open-closed intervals we would get an interval representation I' with intervals in $\mathcal{U}^{\pm,+-}$. \blacksquare

So from now on we assume that u and d are in a same connected component. We also assume, up to translating the whole interval representation, that the intervals for a, b, c, d, e are fixed and that $\ell(a) = 0$. We now explore all the possible values for $\ell(u)$:

- $\ell(u) < -2$: This leads (see the appendix for details) to class \mathcal{A} :

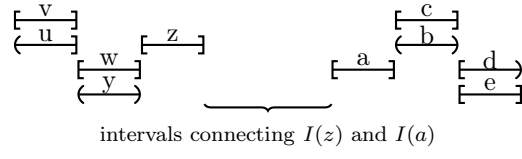


Figure 15: Intervals representation of class \mathcal{A}

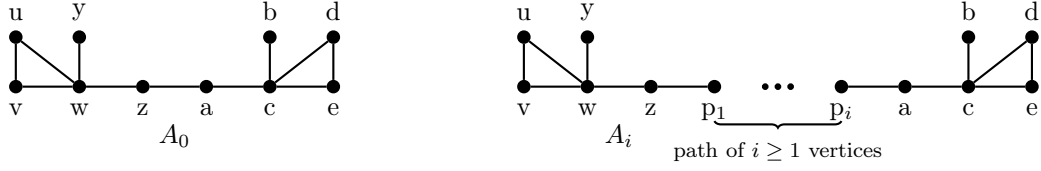


Figure 16: The class \mathcal{A}

- $\ell(u) \geq 3$: This leads to classes \mathcal{B} , \mathcal{B}' and \mathcal{B}'' :

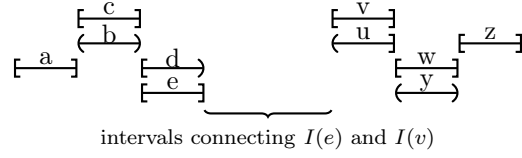


Figure 17: Intervals representation of class \mathcal{B}

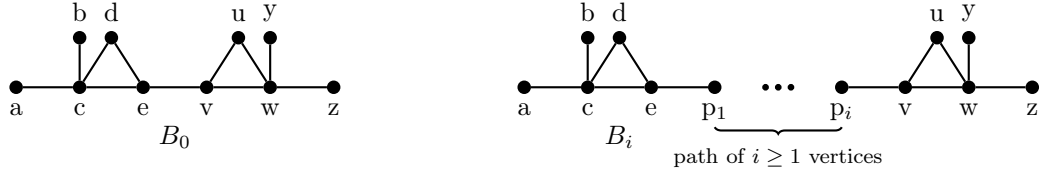


Figure 18: The class \mathcal{B}

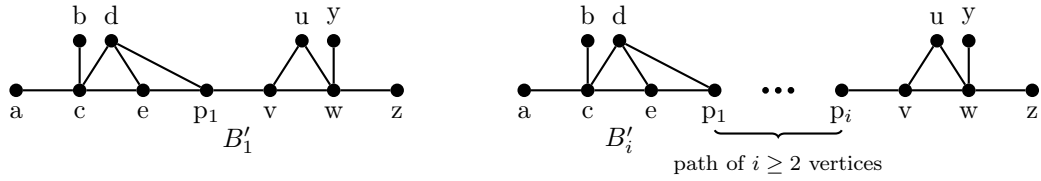


Figure 19: The class \mathcal{B}'

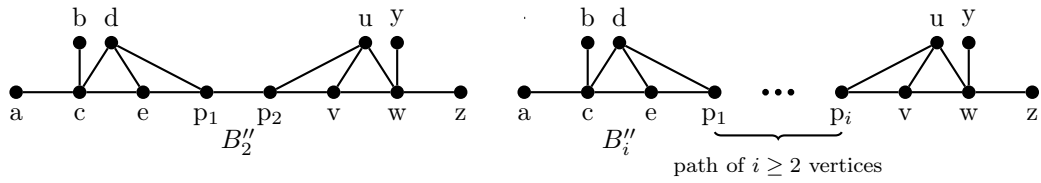


Figure 20: The class \mathcal{B}''

- $\ell(u) \in \mathbb{Z}$ and $-2 \leq \ell(u) < 3$:

► $\ell(u) = -2$:



Figure 21: The graph C_{-2}

► $\ell(u) = -1$:



Figure 22: The graph C_{-1}

► $\ell(u) = 0$:



Figure 23: The graph C_0

► $\ell(u) = 1$:



Figure 24: The graph C_1

► $\ell(u) = 2$:

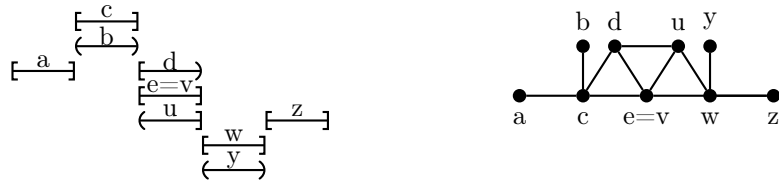


Figure 25: The graph C_2

- $-2 < \ell(u) < 3$ and $\ell(u) \notin \mathbb{Z}$:

► $-2 < \ell(u) < -1$:

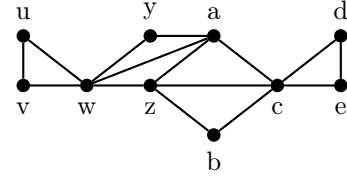
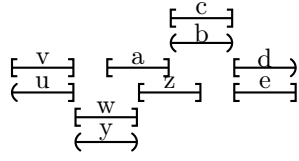


Figure 26: The graph C'_{-2}

► $-1 < \ell(u) < 0$:

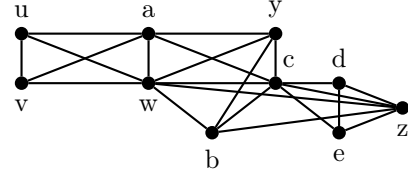
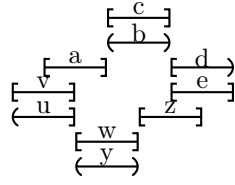


Figure 27: The graph C'_{-1}

► $0 < \ell(u) < 1$:

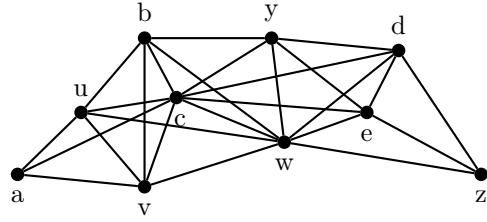
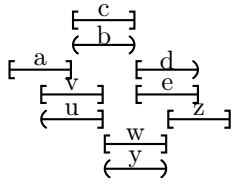


Figure 28: The graph C'_0

► $1 < \ell(u) < 2$:

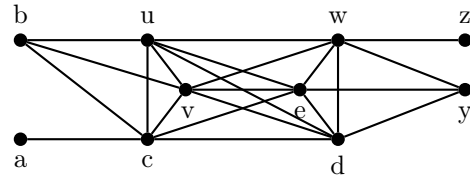
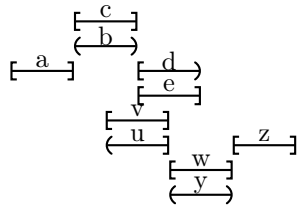


Figure 29: The graph C'_1

► $2 < \ell(u) < 3$:

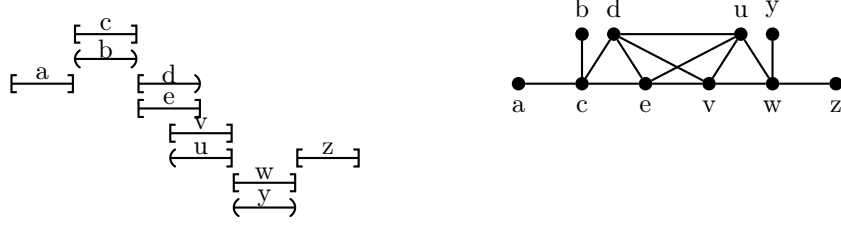
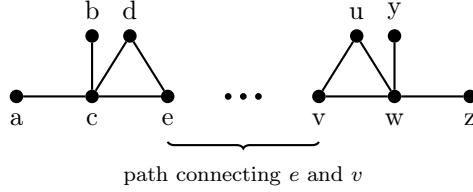


Figure 30: The graph C'_2

We have to add the graphs which are forbidden even for \mathcal{G} . From the class \mathcal{R} we only need R_0 and R_1 since the other ones are supergraphs of graphs in \mathcal{B} . We need $K_{2,3}^*$ and all the graphs in \mathcal{S} and \mathcal{S}' . Finally, from the class \mathcal{T} we only have to add the graphs $T_{0,j}$ for $j \geq 0$ and $T_{1,1}$ because the $T_{i,j}$ with $i > 1$ and $j > 1$ are supergraphs of graphs in \mathcal{B} , the $T_{1,j}$ for $j > 0$ are supergraphs of graphs in \mathcal{B}' and because for every $i, j \geq 0$, $T_{i,j} \simeq T_{j,i}$.

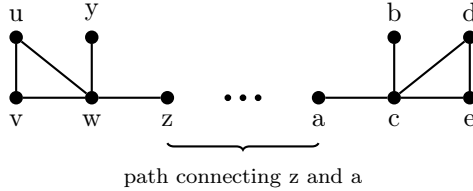
Now we check that all these graphs are indeed forbidden. Since $\mathcal{G}^{\pm,+} \subset \mathcal{G}$, we only need to check the classes we introduce in this article: \mathcal{A} , \mathcal{B} , \mathcal{B}' , \mathcal{B}'' , \mathcal{C} and \mathcal{C}' .

First, we justify the fact that the classes \mathcal{B} , \mathcal{B}' and \mathcal{B}'' are forbidden. This is because the graphs in these classes contain the following pattern:



Indeed, Lemma 1 specifies that the two copies of $K_{1,4}^*$ must be represented, up to symmetry, as in Figure 9. Since there is a path between e and v , which is vertex-disjoint from d and u , the two interval representations must be symmetrical, hence the need for the two types of semi-closed intervals.

For the class \mathcal{A} , we have the following pattern:



Here again we must have two occurrences of the interval configuration shown in Figure 9, but here vertices a and z are connected by a path which is vertex-disjoint from the two $K_{1,4}^*$, so these two occurrences must be symmetrical, hence these graphs are forbidden.

For the graphs C'_{-2} , C'_{-1} , C'_0 , C'_1 and C'_2 the point is that we have two vertex-disjoint $K_{1,4}^*$ (*decba* and *uvwyz*). By Lemma 1 we know that they can be represented by only two sets of intervals. However if we

begin to draw the intervals for $decba$, then there is only one choice for $uvwyz$, up to a small translation. For the graphs C_{-2} , C_{-1} , C_0 , C_1 and C_2 the argument is the same, except that the two $K_{1,4}^*$ share some vertices. We first begin to draw $decba$, and then realize that the other intervals must be exactly as in the above figures.

From what precedes we can state:

Theorem 7. *A graph G is in $\mathcal{G}^{\pm,+}$ if and only if it is a $\mathcal{A} \cup \mathcal{B} \cup \mathcal{B}' \cup \mathcal{B}'' \cup \mathcal{C} \cup \mathcal{C}' \cup \mathcal{S} \cup \mathcal{S}' \cup \{T_{0,j} : j \geq 0\} \cup \{T_{1,1}, R_0, R_1, K_{2,3}^*\}$ -free interval graph.*

Furthermore:

Theorem 8. *The graphs of Theorem 7 are minimal forbidden induced subgraphs for the class $\mathcal{G}^{\pm,+}$.*

Proof. We already proved that these graphs are forbidden, we now only need to prove that they are minimal with this respect.

For the graphs introduced in this section ($\mathcal{A}, \mathcal{B}, \mathcal{B}', \mathcal{B}'', \mathcal{C}$ and \mathcal{C}'), the proof is rather straightforward. We only need to show that if we remove any vertex the resulting graph is no longer forbidden.

If we remove a “ p_i ” vertex in one path, then we disconnect the graph, and can take the symmetry of one of the two components, in terms of interval representation, so as not to have the two different types of semi-closed intervals. If we remove another vertex, then it is easy to see, through the interval representations given above, or more directly from Lemma 2, that the graph is no longer forbidden: we can shift some intervals and close one type of semi-closed intervals.

Now let us consider the graphs in \mathcal{S} , \mathcal{S}' , $T_{0,j}$ for $j \geq 0$, $T_{1,1}$, R_0 and R_1 . It is immediate that $K_{2,3}^*$, R_0 , R_1 and $T_{1,1}$ are minimal.

We then define $\mathcal{O} = \mathcal{S} \cup \mathcal{S}' \cup \{T_{0,j} : j \geq 0\}$. For the graphs in \mathcal{O} , we know by [3] that they are minimal for the class \mathcal{G} . But from what precedes, if we know that a graph G belongs to \mathcal{G} , then $\mathcal{A} \cup \mathcal{B} \cup \mathcal{B}' \cup \mathcal{B}'' \cup \mathcal{C} \cup \mathcal{C}'$ is a minimal set of forbidden induced subgraph for G to belong to $\mathcal{G}^{\pm,+}$ (given the fact that $G \in \mathcal{G}$). So it is sufficient to show that no graph in \mathcal{O} admits as induced subgraph a graph in the previous list.

First, note that the graphs in the classes $\mathcal{A}, \mathcal{B}, \mathcal{B}'$ and \mathcal{B}'' are not induced subgraphs of graphs in \mathcal{O} because the former contain two disjoint copies of $K_{1,4}^*$ and from the latter only $T_{0,j}$ also contain two such copies, but the rest of the graph does not match (in our lists of graphs, y and w are not adjacent). The same kind of argument applies to graphs C_{-2} and C_{-1} . C_1 has two degree 5 vertices, which no graph in \mathcal{O} contains. C_0 and C'_{-2} both contain one degree five vertex which could correspond to only a few vertices in \mathcal{S} and \mathcal{S}' , but we can see that none of this places matches the rest of our graphs. Now C'_{-1} , C'_0 and C'_1 have vertices of degree at least 6 which does not appear in \mathcal{O} . Finally C'_2 contains a K_4 , which only appears in \mathcal{S}' but it is easy to see that C'_2 is not an induced subgraph of any graph of \mathcal{S}' . ■

3.3 Algorithmical perspectives

The proof of the previous theorem leads to some algorithms enabling us to detect if a graph is in $\mathcal{U}^{\pm,+}$ and if it is, to give a corresponding interval representation of it, as stated below. The following theorems use the standard notations $n = |V|$ and $m = |E|$.

Theorem 9. *There exists an algorithm which, given a $\mathcal{U}^{\pm,+}$ -graph G , produces a $\mathcal{U}^{\pm,+}$ -representation of G in time $O(n^2 + m)$.*

Proof. We give here the algorithm, which takes a graph G as input:

1. Prune G into a twin-free graph G' ;
2. Get a \mathcal{U} -representation of it;
3. For each connected component:
 - a. From right to left, try to close every open-closed interval with the transformations of the proof of Lemma 2;
 - b. Try to close similarly every open-closed interval, from left to right;
4. Symmetrise the interval representation of the connected component which contain some open-closed intervals;
5. Return the obtained interval representation.

First, we claim that the algorithm is correct. Indeed, since the input graph is a $\mathcal{U}^{\pm,+}$ -graph, we know that in each connected component we can close all semi-closed interval of one type. We use the transformations of the proof of Lemma 2, which work if the semi-closed interval we try to close is not in a peculiar neighborhood of intervals. The main point is that by performing our transformations, given the direction of the sweeps, we do not create new possible transformations, so when we try to close any semi-closed interval, if we fail then we know that it will never be possible to close it. We may also note that in the figure of Lemma 2, in our case $I(z)$ may be a closed-open interval and $I(a)$ may be an open-closed interval, in both cases they may prevent us from closing the associated $I(u)$ or $I(d)$. In case we have a specific neighborhood of intervals as one of the two in Lemma 2, we showed that it cannot be closed, hence the correctness of the algorithm. Note that because there could be a succession of open-closed intervals as in the proof of Theorem 8 (see Claim 3), we need to try to close them from right to left.

Concerning the time complexity, operation 1 can be done in time $O(n + m)$ as in [4]. Operation 2 can be done in $O(n^2 + m)$ as shown in [8]. Operation 3 takes time $O(n^2)$: we try to close each interval at most once, and trying to closing an interval takes $O(n)$ if we have to shift many intervals, or $O(1)$ (we check only the existence of 4 intervals at specific positions). Operation 4 takes time $O(n)$, hence the overall quadratic complexity, given the fact that the graphs we deal with are simple. ■

From this algorithm we can derive another one to test if a graph is in $\mathcal{U}^{\pm,+}$, but first we state a simple lemma about the recognition of the class of mixed unit intervals. This lemma comes easily from the Algorithm 17 presented in [8].

Proposition 1. *The class of mixed unit interval graphs can be recognized in time $O(n^2)$.*

Proof. The proof of this result comes from the Algorithm 17 in [8] that we used in the previous proof. Although it is not mentioned in the paper introducing it, it is possible to take any graph G as input for it. Then either the algorithm runs as expected and the obtained interval representation is correct (which we can check in linear time), or the algorithm fails at some point. Indeed, the algorithm is made of a loop of at most n sweeps, each of which taking $O(n)$ -time, and it is possible to check that the algorithm runs correctly without worsening the complexity of the algorithm. ■

Theorem 10. *The class of almost-mixed unit interval graphs can be recognized in time $O(n^2)$.*

Proof. The proof is straightforward: we first apply the algorithm described in the proof of the previous theorem, using Proposition 1. If the interval representation we get at the end of the algorithm contains at most one type of semi-closed intervals, then this proves that our graph is an almost-mixed unit interval graph. Else, the interval representation contains two types of semi-mixed intervals. As we showed that our algorithm is correct, then this incorrect result means that the graph is not in $\mathcal{U}^{\pm,+-}$. Also, by construction of the algorithm, in case the input graph is a \mathcal{U} -graph, the output of our algorithm gives a certificate that the graph is not a $\mathcal{U}^{\pm,+-}$: the representation must contain in a same connected component two neighborhoods of intervals like $abcde$ and $uvwxyz$ as in Lemma 2.

Testing if the output of the algorithm is of the right form can be done in time $O(n)$, hence the claimed complexity. ■

4 Appendix

We explain here how we get the classes \mathcal{A} , \mathcal{B} , \mathcal{B}' and \mathcal{B}'' of section 3.

Class \mathcal{A} corresponds to the graphs built in the case when $\ell(u) < -2$. The intervals representing the p_i 's only intersects one another, plus $I(z)$ and $I(a)$: we do not consider other vertices that could intersect $I(w)$ for instance, since we look for minimal forbidden subgraphs. So we get a graph which looks like this, if $i = 3$:

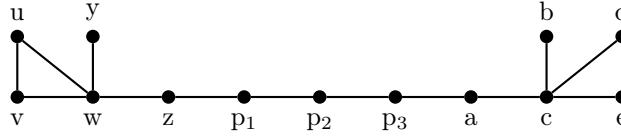


Figure 31: The graph \mathcal{A}_3

Now there could be edges between, for instance, z and p_2 , or between p_1 and p_3 . But in this case, we would remove p_1 in the first case, or p_2 in the second one, and get an induced \mathcal{A}_2 .

Now we explain how we get the remaining classes. They come from the case $\ell(u) \geq 3$. From the same remarks as in the previous case, we get a graph like this one:

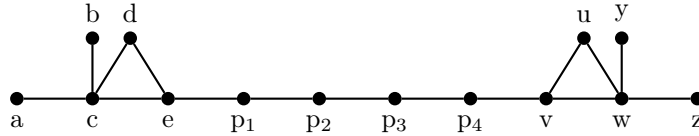


Figure 32: The graphs \mathcal{B}_4

For the same reasons as for class \mathcal{A} , since we only want *minimal* forbidden induced subgraphs, the p_i 's cannot intersect one another, and we only consider possible neighbors for the vertices with “extremal” intervals, that is v and e . Also, we can see in the interval representation that every neighbor of d is also a neighbor of e , and every neighbor of u is also a neighbor of v . Since, following the same arguments as for class \mathcal{A} , we discard the graphs where e is connected to a p_i with $i > 1$ or where v is connected to a

p_i which is not the last one, the same applies for the neighbors of d and u . Thus we only get the three classes \mathcal{B} , \mathcal{B}' and \mathcal{B}'' .

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